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Title: Seidl: Programmoptimierung (28.10.2013)

Date: Mon Oct 28 14:01:08 CET 2013

Duration: 88:10 min

Pages: 48

We are looking for solutions for systems of constraints of the form:

$$x_i \quad \sqsupseteq \quad f_i(x_1, \ldots, x_n) \qquad\qquad (*)$$

where:

x_i	unknown	here:	$\mathcal{A}[u]$
$\mathbb{D}$	values	here:	2^{Expr}
$\sqsubseteq \ \subseteq \ \mathbb{D} \times \mathbb{D}$	ordering relation	here:	$\supseteq$
$f_i : \mathbb{D}^n \to \mathbb{D}$	constraint	here:	...

Constraint for $\mathcal{A}[v]$ $(v \neq start)$:

$$\mathcal{A}[v] \quad \subseteq \quad \bigcap\{[\![k]\!]^\sharp(\mathcal{A}[u]) \mid k = (u, \_, v) \text{ edge}\}$$

Because:

$$x \sqsupseteq d_1 \wedge \ldots \wedge x \sqsupseteq d_k \quad \text{iff} \quad x \sqsupseteq \bigsqcup\{d_1, \ldots, d_k\}$$:-)

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(2) $\mathbb{D}_1 = \mathbb{D}_2 = \mathbb{Z}$ (with the ordering "$\leq$"). Then:

- $\mathsf{inc}\,x = x + 1$ is monotonic.
- $\mathsf{dec}\,x = x - 1$ is monotonic.

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(2) $\mathbb{D}_1 = \mathbb{D}_2 = \mathbb{Z}$ (with the ordering "$\leq$"). Then:

- $\mathsf{inc}\,x = x + 1$ is monotonic.
- $\mathsf{dec}\,x = x - 1$ is monotonic.
- $\mathsf{inv}\,x = -x$ is not monotonic :-)

Theorem:

If $f_1 : \mathbb{D}_1 \to \mathbb{D}_2$ and $f_2 : \mathbb{D}_2 \to \mathbb{D}_3$ are monotonic, then also $f_2 \circ f_1 : \mathbb{D}_1 \to \mathbb{D}_3$:-)

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Theorem:

If $\mathbb{D}_2$ is a complete lattice, then the set $[\mathbb{D}_1 \to \mathbb{D}_2]$ of monotonic functions $f : \mathbb{D}_1 \to \mathbb{D}_2$ is also a complete lattice where

$$f \sqsubseteq g \quad \text{iff} \quad f\,x \sqsubseteq g\,x \quad \text{for all } x \in \mathbb{D}_1$$

$\bigsqcup \mathfrak{F} = g$

$g\,x = \bigsqcup \{f\,x \mid f \in \mathfrak{F}\}$

For functions $f_i\,x = a_i \cap x \cup b_i$, the operations "$\circ$", "$\sqcup$" and "$\sqcap$" can be explicitly defined by:

$$\begin{array}{lll}
(f_2 \circ f_1)\,x & = & \boxed{a_1 \cap a_2} \cap x \cup \boxed{a_2 \cap b_1 \cup b_2} \\
(f_1 \sqcup f_2)\,x & = & \boxed{(a_1 \cup a_2)} \cap x \cup \boxed{b_1 \cup b_2} \\
(f_1 \sqcap f_2)\,x & = & \boxed{(a_1 \cup b_1) \cap (a_2 \cup b_2)} \cap x \cup \boxed{b_1 \cap b_2}
\end{array}$$

$$(f_2 \circ f_1)\,x = a_2 \cap (a_1 \cap x \cup b_1) \cup b_2$$
$$= \boxed{a_1 \cap a_2} \cap x \cup \boxed{a_2 \cap b_1 \cup b_2}$$

Wanted: minimally small solution for:

$$x_i \sqsupseteq f_i(x_1, \ldots, x_n), \quad i = 1, \ldots, n \qquad (*)$$

where all $f_i : \mathbb{D}^n \to \mathbb{D}$ are monotonic.

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Idea:

- Consider $\boxed{F} : \mathbb{D}^n \to \mathbb{D}^n$ where
  $F(x_1, \ldots, x_n) = (y_1, \ldots, y_n)$ with $y_i = f_i(x_1, \ldots, x_n)$.

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- If all f_i are monotonic, then also F :-)

$$(x_1, \ldots, x_n) \sqsubseteq (y_1, \ldots, y_n)$$
$$\Longleftrightarrow \forall j.\ x_j \sqsubseteq y_j$$

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- We successively approximate a solution. We construct:

$$\underline{\perp}, \quad F\,\underline{\perp}, \quad F^2\,\underline{\perp}, \quad F^3\,\underline{\perp}, \quad \ldots$$

Hope: We eventually reach a solution ... ???

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$$x \sqsupseteq F\,x$$

$$(x \in \mathbb{D}^n)$$

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Example: $\mathbb{D} = 2^{\{a,b,c\}}, \quad \sqsubseteq \, = \, \subseteq$

$$\begin{aligned} x_1 &\supseteq \{a\} \cup x_3 \\ x_2 &\supseteq x_3 \cap \{a, b\} \\ x_3 &\supseteq x_1 \cup \{c\} \end{aligned}$$

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$$\begin{aligned} x_1 &\supseteq \{a\} \cup x_3 \; \emptyset \\ x_2 &\supseteq \cancel{x_3}\,\emptyset \cap \{a, b\} \\ x_3 &\supseteq \cancel{x_1}\,\emptyset \cup \{c\} \end{aligned}$$

The Iteration:

	0	1	2	3	4
x_1	$\emptyset$	a			
x_2	$\emptyset$	$\emptyset$			
x_3	$\emptyset$	{c}			

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$$\begin{aligned} x_1 &\supseteq \{a\} \cup \cancel{x_3}\, c \\ x_2 &\supseteq c\,\cancel{x_3} \cap \{a, b\} \\ x_3 &\supseteq a\,\cancel{x_1} \cup \{c\} \end{aligned}$$

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Theorem

- $\bot, F\,\bot, F^2\,\bot, \ldots$ form an ascending chain :

$$\bot \sqsubseteq F\,\bot \sqsubseteq F^2\,\bot \sqsubseteq \ldots$$

- If $F^k\,\bot = F^{k+1}\,\bot$, a solution is obtained which is the least one :-)
- If all ascending chains are finite, such a k always exists.

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$\forall i.\ F^i\,\bot \sqsubseteq F^{i+1}\,\bot$

Proof

The first claim follows by complete induction:

Foundation: $F^0\,\bot = \bot \sqsubseteq F^1\,\bot$:-)

Step: Assume $F^{i-1}\,\bot \sqsubseteq F^i\,\bot$. Then

$$F^i\,\bot = F\,(F^{i-1}\,\bot) \sqsubseteq F\,(F^i\,\bot) = F^{i+1}\,\bot$$

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If $\mathbb{D}$ is finite, a solution can be found which is definitely the least :-)

Question:

What, if $\mathbb{D}$ is not finite ???

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- $\perp, F\,\perp, F^2\,\perp, \ldots$ form an ascending chain :

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Theorem Knaster – Tarski

Assume $\mathbb{D}$ is a complete lattice. Then every monotonic function $f : \mathbb{D} \to \mathbb{D}$ has a least fixpoint $d_0 \in \mathbb{D}$.

Let $P = \{d \in \mathbb{D} \mid f\,d \sqsubseteq d\}$.

Then $d_0 = \bigsqcap P$.

Bronisław Knaster (1893-1980), topology

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Then $d_0 = \bigsqcap P$.

Proof:

(1) $d_0 \in P$:

$f\,d_0 \sqsubseteq f\,d \sqsubseteq d$ for all $d \in P$

$\Longrightarrow$ $f\,d_0$ is a lower bound of P

$\Longrightarrow$ $f\,d_0 \sqsubseteq d_0$ since $d_0 = \bigsqcap P$

$\Longrightarrow$ $d_0 \in P$:-)

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$\boxed{f\,d_0 \sqsubseteq d_0}$ by (1)

$\Longrightarrow$ $f(\boxed{f\,d_0}) \sqsubseteq \boxed{f\,\boxed{d_0}}$ by monotonicity of f

$\Longrightarrow$ $f\,d_0 \in P$

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(3) $\quad d_0$ is least fixpoint:

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(3) $\quad d_0$ is least fixpoint:

$f\,d_1 = d_1 \sqsubseteq d_1$ an other fixpoint

$\Longrightarrow$ $d_1 \in P$

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The least fixpoint d_0 is in P and a lower bound :-)

$\Longrightarrow$ d_0 is the least value x with $x \sqsupseteq f\,x$

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Application:

Assume $\qquad x_i \sqsupseteq f_i(x_1, \ldots, x_n), \quad i = 1, \ldots, n \qquad (*)$

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$\Longrightarrow$ least solution of $(*)$ $==$ least fixpoint of F :-)

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Example 2: $\mathbb{D} = \mathbb{N} \cup \{\infty\}$

Assume $f\,x = x + 1$. Then

$$f^i \perp = f^i\,0 = i \quad \sqsubset \quad i + 1 = f^{i+1} \perp$$

Conclusion:

Systems of inequations can be solved through fixpoint iteration, i.e., by repeated evaluation of right-hand sides :-)

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$\Longrightarrow$ Ordinary iteration will never reach a fixpoint :-(

$\Longrightarrow$ Sometimes, transfinite iteration is needed :-)

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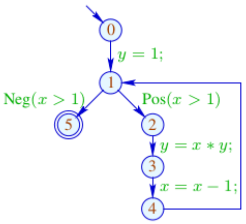

0	
1	
2	
3	
4	
5	

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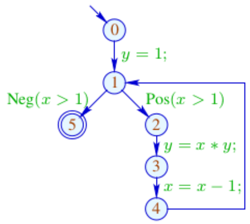

	1
0	$\emptyset$
1	$\{1, x > 1, x - 1\}$
2	$Expr$
3	$\{1, x > 1, x - 1\}$
4	$\{1\}$
5	$Expr$

~~Expr~~ $\{1, x > 1, x - 1\}$

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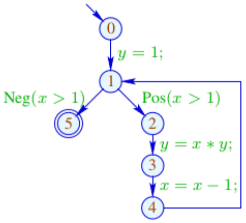

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0	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	
1	$\{1, x > 1, x - 1\}$	$\{1\}$	$\{1\}$	$\{1\}$	
2	$Expr$	$\{1, x > 1, x - 1\}$	$\{1, x > 1\}$	$\{1, x > 1\}$	
3	$\{1, x > 1, x - 1\}$	$\{1, x > 1, x - 1\}$	$\{1, x > 1, x - 1\}$	$\{1, x > 1\}$	dito
4	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	
5	$Expr$	$\{1, x > 1, x - 1\}$	$\{1, x > 1\}$	$\{1, x > 1\}$	