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Title: Seidl: Programmoptimierung (09.01.2013)

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Discussion:

- Solutions only matter within the bounds to the iteration variables.
- Every integer solution there provides a conflict.
- Fusion of loops is possible if no conflicts occur :-)
- The given special case suffices to solve the case one variable over $\mathbb{Z}$:-)
- The number of variables in the inequations corresponds to the nesting-depth of for-loops $\Longrightarrow$ in general, is quite small :-)

Discussion:

- Integer Linear Programming (ILP) can decide satisfiability of a finite set of equations/inequations over $\mathbb{Z}$ of the form:

$$\sum_{i=1}^{n} a_i \cdot x_i = b \quad \text{bzw.} \quad \sum_{i=1}^{n} a_i \cdot x_i \geq b\,, \quad a_i \in \mathbb{Z}$$

- Moreover, a (linear) cost function can be optimized :-)
- Warning: The decision problem is in general, already NP-hard !!!
- Notwithstanding that, surprisingly efficient implementations exist.
- Not just loop fusion, but also other re-organizations of loops yield ILP problems ...

Background 5: Presburger Arithmetic

Many problems in computer science can be formulated without multiplication :-)

Let us first consider two simple special cases ...

1. Linear Equations

$$\begin{array}{rcrcrcr} 2x & + & 3y & & & = & 24 \\ x & - & y & + & 5z & = & 3 \end{array}$$

Question:

- Is there a solution over $\mathbb{Q}$?
- Is there a solution over $\mathbb{Z}$?
- Is there a solution over $\mathbb{N}$?

Let us reconsider the equations:

$$\begin{array}{rcrcrcr} 2x & + & 3y & & & = & 24 \\ x & - & y & + & 5z & = & 3 \end{array}$$

Answers:

- Is there a solution over $\mathbb{Q}$? Yes
- Is there a solution over $\mathbb{Z}$? No
- Is there a solution over $\mathbb{N}$? No

Complexity:

- Is there a solution over $\mathbb{Q}$? Polynomial
- Is there a solution over $\mathbb{Z}$? Polynomial
- Is there a solution over $\mathbb{N}$? NP-hard

Solution Method for Integers:

Observation 1:

$$a_1 x_1 + \ldots + a_k x_k = b \qquad (\forall i: \ a_i \neq 0)$$

has a solution iff

$$\gcd\{a_1, \ldots, a_k\} \ \mid \ b$$

Example:

$$5y - 10z = 18$$

has no solution over $\mathbb{Z}$:-)

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Observation 2:

Adding a multiple of one equation to another does not change the set of solutions :-)

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$$\Longrightarrow$$

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Observation 3:

Adding multiples of columns to another column is an invertible transformation which we keep track of in a separate matrix ...

$$\begin{array}{ccc|rcrcrcr} 1 & 0 & 0 & & & 5y & - & 10z & = & 18 \\ 0 & 1 & 0 & x & - & y & + & 5z & = & 3 \\ 0 & 0 & 1 & & & & & & & \end{array}$$

$$\Longrightarrow$$

$$\begin{array}{ccc|rcrcrcr} 1 & 0 & 0 & & & 5y & & & = & 18 \\ 0 & 1 & 2 & x & - & y & + & 3z & = & 3 \\ 0 & 0 & 1 & & & & & & & \end{array}$$

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Adding multiples of columns to another column is an invertible transformation which we keep track of in a separate matrix ...

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$$\Longrightarrow$$

$$\begin{array}{ccc|cccccc} 1 & 0 & -3 & & & 5y & = & 18 \\ 0 & 1 & 2 & x & - & y & = & 3 \\ 0 & 0 & 1 & & & & & \end{array}$$

$\Longrightarrow$ triangular form !!

Observation 4:

- A special solution of a triangular system can be directly read off :-)
- All solutions of a homogeneous triangular system can be directly read off :-)
- All solutions of the original system can be recovered from the solutions of the triangular system by means of the accumulated transformation matrix:-))

Solving over $\mathbb{N}$

- ... is of major practical importance;
- ... has led to the development of many new techniques;
- ... easily allows to encode NP-hard problems;
- ... remains difficult if just three variables are allowed per equation.

2. One Polynomial Special Case:

$$\begin{array}{rcl} x & \geq & y+5 \\ 19 \geq & x & \\ y & \geq & 13 \\ y & \geq & x-7 \end{array}$$

- There are at most 2 variables per in-equation;
- no scaling factors.

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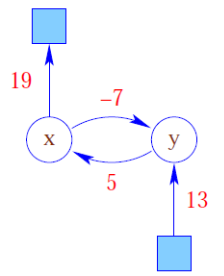
Idea: Represent the system by a graph:



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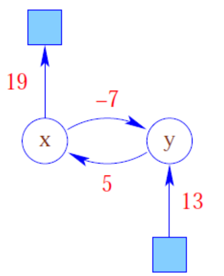

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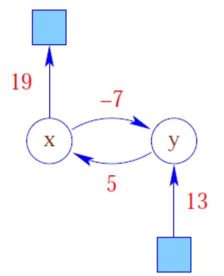

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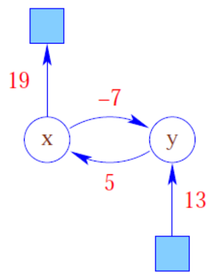

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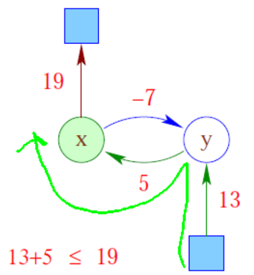

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Compute the reflexive and transitive closure of the edge weights!

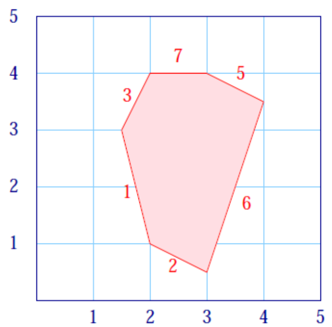
3. A General Solution Method:

Idea: Fourier-Motzkin Elimination

- Successively remove individual variables x !
- All in-equations with positive occurrences of x yield lower bounds.
- All in-equations with negative occurrences of x yield upper bounds.
- All lower bounds must be at most as big as all upper bounds ;-))

Example:

$$\begin{array}{rcll} 9 & \leq & 4x_1 + x_2 & (1) \\ 4 & \leq & x_1 + 2x_2 & (2) \\ 0 & \leq & 2x_1 - x_2 & (3) \\ 6 & \leq & x_1 + 6x_2 & (4) \\ -11 & \leq & -x_1 - 2x_2 & (5) \\ -17 & \leq & -6x_1 + 2x_2 & (6) \\ -4 & \leq & -x_2 & (7) \end{array}$$



For x_1 we obtain:

$$\begin{array}{rcll} 9 & \leq & 4x_1 + x_2 & (1) \\ 4 & \leq & x_1 + 2x_2 & (2) \\ 0 & \leq & 2x_1 - x_2 & (3) \\ 6 & \leq & x_1 + 6x_2 & (4) \\ -11 & \leq & -x_1 - 2x_2 & (5) \\ -17 & \leq & -6x_1 + 2x_2 & (6) \\ -4 & \leq & -x_2 & (7) \end{array} \qquad \begin{array}{rcll} \frac{9}{4} - \frac{1}{4}x_2 & \leq & x_1 & (1) \\ 4 - 2x_2 & \leq & x_1 & (2) \\ \frac{1}{2}x_2 & \leq & x_1 & (3) \\ 6 - 6x_2 & \leq & x_1 & (4) \\ x_1 & \leq & 11 - 2x_2 & (5) \\ x_1 & \leq & \frac{17}{6} + \frac{1}{3}x_2 & (6) \\ -4 & \leq & -x_2 & (7) \end{array}$$

If such an x_1 exists, all lower bounds must be bounded by all upper bounds, i.e.,

$$\begin{array}{rcllcrcll}
\frac{9}{4}-\frac{1}{4}x_2 & \leq & 11-2x_2 & (1,5) & & -35 & \leq & -7x_2 & (1,5)\\
\frac{9}{4}-\frac{1}{4}x_2 & \leq & \frac{17}{6}+\frac{1}{3}x_2 & (1,6) & & -\frac{7}{12} & \leq & \frac{7}{12}x_2 & (1,6)\\
4-2x_2 & \leq & 11-2x_2 & (2,5) & & -7 & \leq & 0 & (2,5)\\
4-2x_2 & \leq & \frac{17}{6}+\frac{1}{3}x_2 & (2,6) & & \frac{7}{6} & \leq & \frac{7}{3}x_2 & (2,6)\\
\frac{1}{2}x_2 & \leq & 11-2x_2 & (3,5) & \text{or} & -22 & \leq & -5x_2 & (3,5)\\
\frac{1}{2}x_2 & \leq & \frac{17}{6}+\frac{1}{3}x_2 & (3,6) & & -\frac{17}{6} & \leq & -\frac{1}{6}x_2 & (3,6)\\
6-6x_2 & \leq & 11-2x_2 & (4,5) & & -5 & \leq & 4x_2 & (4,5)\\
6-6x_2 & \leq & \frac{17}{6}+\frac{1}{3}x_2 & (4,6) & & \frac{19}{6} & \leq & \frac{19}{3}x_2 & (4,6)\\
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\end{array}$$

This is the one-variable case which we can solve exactly:

$$\max\left\{-1, \boxed{\tfrac{1}{2}}, -\tfrac{5}{4}, \tfrac{1}{2}\right\} \quad\leq\quad x_2 \quad\leq\quad \min\left\{5, \tfrac{22}{5}, 17, \boxed{4}\right\}$$

From which we conclude: $x_2 \in [\frac{1}{2}, 4]$:-)

In General:

- The original system has a solution over $\mathbb{Q}$ iff the system after elimination of one variable has a solution over $\mathbb{Q}$:-)
- Every elimination step may square the number of in-equations
  $\Longrightarrow$ exponential run-time :-((
- It can be modified such that it also decides satisfiability over $\mathbb{Z}$
  $\Longrightarrow$ Omega Test

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Idea:

- We successively remove variables. Thereby we omit division ...
- If x only occurs with coefficient ± 1, we apply Fourier-Motzkin elimination :-)
- Otherwise, we provide a bound for a positive multiple of x ...

Consider, e.g., (1) and (6) :

$$\begin{array}{rcl} 6 \cdot x_1 & \leq & 17 + 2x_2 \\ 9 - x_2 & \leq & 4 \cdot x_1 \end{array}$$

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W.l.o.g., we only consider strict in-equations:

$$\begin{aligned} 6 \cdot x_1 &< 18 + 2x_2 \\ 8 - x_2 &< 4 \cdot x_1 \end{aligned}$$

... where we always divide by gcds:

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We thereby obtain:

- If one derived in-equation is unsatisfiable, then also the overall system :-)
- If all derived in-equations are satisfiable, then there is a solution which, however, need not be integer :-(
- An integer solution is guaranteed to exist if there is sufficient separation between lower and upper bound ...
- Assume $\alpha < a \cdot x \qquad b \cdot x < \beta$.
  Then it should hold that:
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  and moreover:
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$$12 \quad < \quad 4 \cdot (9 + x_2) - 3 \cdot (8 - x_2)$$

or:

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In the example, also these strengthened in-equations are satisfiable

$\Longrightarrow$ the system has a solution over $\mathbb{Z}$:-)

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If the system is solvable, the test generally terminates rapidly.

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