Script generated by TTT

Title: Nipkow: Info2 (12.11.2013)

Date: Tue Nov 12 15:32:23 CET 2013

Duration: 86:03 min

Pages: 136



Example: proof by cases





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rem x (y:ys) | x==y = rem x ys
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To show: rem z ((x:xs)++ys) = rem z (x:xs) ++ rem z ys
| Proof by cases:
| Case z == x:
| rem z ((x:xs) ++ ys) = rem z (xs ++ ys) = rem z (xs ++ ys) = -- by def of ++ and rem = rem z xs ++ rem z ys --- by IH
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     = x : rem z (xs ++ ys)
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    rem z (x:xs) ++ rem z ys
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    Case z \neq x:
    rem z ((x:xs) ++ ys)
    = x : rem z (xs ++ ys)
                                -- by def of ++ and rem
     = x : (rem z xs ++ rem z ys) -- by IH
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    Induction step:
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    Case z \neq x:
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     = x : rem z (xs ++ ys)
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     = x : (rem z xs ++ rem z ys) -- by IH
     rem z (x:xs) ++ rem z ys
     = x : (rem z xs ++ rem z ys) -- by def of rem and ++
```



Proof by cases



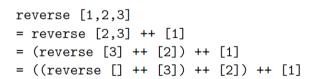
Inefficiency of reverse

reverse [1,2,3]

Works just as well for ${\tt if-then-else}$, for example



Inefficiency of reverse





Inefficiency of reverse

```
reverse [1,2,3]
= reverse [2,3] ++ [1]
= (reverse [3] ++ [2]) ++ [1]
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= ([3] ++ [2]) ++ [1]
= (3 : ([] ++ [2])) ++ [1]
= [3,2] ++ [1]
```



Inefficiency of reverse



An improvement: itrev

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= ([3] ++ [2]) ++ [1]
= (3 : ([] ++ [2])) ++ [1]
= [3,2] ++ [1]
= 3 : ([2] ++ [1])
= 3 : (2 : ([] ++ [1]))
= [3,2,1]
```



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itrev :: [a] -> [a] -> [a]
itrev [] xs = xs
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itrev [1,2,3] []
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= itrev [] [3,2,1]



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Proof attempt



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Lemma itrev xs [] = reverse xs
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Proof by structural induction on xs
Induction step fails:
To show: itrev (x:xs) [] = reverse xs
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Proof by structural induction on xs
Induction step fails:
To show: itrev (x:xs) [] = reverse xs
  itrev (x:xs) []
  = itrev xs [x] -- by def of itrev
```



Proof attempt

Generalization

Lemma itrev xs ys =

Problem: IH not applicable because too specialized: []



Generalization

```
Lemma itrev xs ys = reverse xs ++ ys
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reverse (x:xs) ++ ys
= (reverse xs ++ [x]) ++ ys -- by def of reverse
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Lemma itrev xs ys = reverse xs ++ ys
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reverse (x:xs) ++ ys
= (reverse xs ++ [x]) ++ ys -- by def of reverse
= reverse xs ++ ([x] ++ ys) -- by Lemma app_assoc
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Note: IH is used with x:ys instead of ys



When using the IH, variables may be replaced by arbitrary expressions, only the induction variable must stay fixed.



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Justification: all variables are implicitly \forall -quantified, except for the induction variable.



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Proof by structural induction on xs

Induction step:

Note: IH is used with x:ys instead of ys



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Induction on the length of a list

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qsort :: Ord a => [a] -> [a]
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    where below = [y | y <- xs, y <= x]
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Proof by induction on the length of the argument of qsort.



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Lemma qsort xs is sorted

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Induction step: In the call qsort (x:xs) we have length below
<= length xs < length(x:xs)</pre>



Induction on the length of a list

Lemma qsort xs is sorted

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Therefore qsort below and qsort above are sorted by IH.



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Therefore $\ensuremath{\operatorname{qsort}}$ below and $\ensuremath{\operatorname{qsort}}$ above are sorted by IH.

By construction below contains only elements (<=x).

Therefore qsort below contains only elements (<=x) (proof!).



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Analogously for above and (x<).

Therefore qsort (x:xs) is sorted.



Is that all?



Is that all? Or should we prove something else about sorting?

How about this sorting function?

```
superquicksort _ = []
```



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How about this sorting function?

superquicksort _ = []

Every element should occur as often in the output as in the input!



5.2 Definedness

Simplifying assumption, implicit so far:

No undefined values



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Two kinds of undefinedness:

head [] raises exception f x = f x + 1



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Undefinedness can be handled, too.

But it complicates life



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Many familiar laws no longer hold unconditionally:

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Two examples:

- Not true: head [] head [] = 0
- ullet From the nonterminating definition

$$f x = f x + 1$$

we could conclude that 0 = 1.



Termination



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Restriction:

The proof methods in this chapter assume that all recursive definitions under consideration terminate.



Termination



How to prove termination

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Restriction:

The proof methods in this chapter assume that all recursive definitions under consideration terminate.

Most Haskell functions we have seen so far terminate.

```
Example
reverse [] = []
reverse (x:xs) = reverse xs ++ [x]
```



How to prove termination

Example

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A function $f :: T1 \rightarrow T$ terminates if there is a *measure function* $m :: T1 \rightarrow \mathbb{N}$ such that

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- and for every recursive call f r in t: m p > m r.



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- and for every recursive call f r in t: m p > m r.
- Note:
 - All primitive recursive functions terminate.



How to prove termination

Example

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A function f :: T1 -> T terminates if there is a measure function m :: T1 -> $\mathbb N$ such that

- for every defining equation f p = t
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Note:

- All primitive recursive functions terminate.
- m can be defined in Haskell or mathematics.



More generally: f :: T1 -> ... -> Tn -> T terminates if there is a measure function m :: T1 -> ... -> Tn -> \mathbb{N} such that

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- for every defining equation f p = t
- and for every recursive call f r in t: m p > m r.

Note:

- All primitive recursive functions terminate.
- m can be defined in Haskell or mathematics.
- The conditions above can be refined to take special Haskell features into account, eg sequential pattern matching.



More generally: f :: T1 -> ... -> Tn -> T terminates if there is a measure function m :: T1 -> ... -> Tn -> \mathbb{N} such that

- for every defining equation f p1 ... pn = t
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 m p1 ... pn > m r1 ... rn.



Infinite values

More generally: f :: T1 -> ... -> Tn -> T terminates if there is a measure function m :: T1 -> ... -> Tn -> \mathbb{N} such that

- for every defining equation $f p1 \dots pn = t$
- and for every recursive call f r1 ... rn in t: m p1 ... pn > m r1 ... rn.

Of course, all other functions that are called by f must also terminate.



Infinite values

Haskell allows infinite values, in particular infinite lists.

Example: [1, 1, 1, ...]

Infinite objects must be constructed by recursion:

ones = 1 : ones



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Note:

- By termination of functions we really mean termination on *finite* values.
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Because we restrict to terminating definitions in this chapter, infinite values cannot arise.

Note:

- By termination of functions we really mean termination on *finite* values.
- For example reverse terminates only on finite lists.

This is fine because we can only construct finite values anyway.



How can infinite values be useful? Because of "lazy evaluation".



Exceptions

If we use arithmetic equations like x - x = 0 unconditionally, we can "lose" exceptions:

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In such cases, we can prove equations e1 = e2 that are only partially correct:

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In such cases, we can prove equations e1 = e2 that are only partially correct:

If e1 and e2 do not produce a runtime exception then they evaluate to the same value.



Summary



Summary

• In this chapter everything must terminate

- In this chapter everything must terminate
- This avoids undefined and infinite values



Summary

- In this chapter everything must terminate
- This avoids undefined and infinite values
- This simplifies proofs

6. Higher-Order Functions

```
Recall [Pic is short for Picture]
```

```
alterH :: Pic -> Pic -> Int -> Pic
alterH pic1 pic2 1 = pic1
alterH pic1 pic2 n = beside pic1 (alterH pic2 pic1 (n-1))
alterV :: Pic -> Pic -> Int -> Pic
alterV pic1 pic2 1 = pic1
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Very similar. Can we avoid duplication?
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Very similar. Can we avoid duplication?

alt f pic1 pic2 1 = pic1
alt f pic1 pic2 n = f pic1 (alt f pic2 pic1 (n-1))
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Very similar. Can we avoid duplication?

alt :: (Pic -> Pic -> Pic) -> Pic -> Pic -> Int -> Pic
alt f pic1 pic2 1 = pic1
alt f pic1 pic2 n = f pic1 (alt f pic2 pic1 (n-1))

alterH pic1 pic2 n = alt beside pic1 pic2 n

alterV pic1 pic2 n = alt above pic1 pic2 n
```



Higher-order functions: Functions that take functions as arguments



Higher-order functions: Functions that take functions as arguments

Higher-order functions capture patterns of computation

```
alterH :: Pic -> Pic -> Int -> Pic
alterH pic1 pic2 1 = pic1
alterH pic1 pic2 n = beside pic1 (alterH pic2 pic1 (n-1))

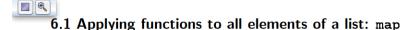
alterV :: Pic -> Pic -> Int -> Pic
alterV pic1 pic2 n = pic1
alterV pic1 pic2 n = above pic1 (alterV pic2 pic1 (n-1))

Very similar. Can we avoid duplication?

alt :: (Pic -> Pic -> Pic) -> Pic -> Pic -> Int -> Pic
alt f pic1 pic2 n = f pic1
alt f pic1 pic2 n = f pic1 (alt f pic2 pic1 (n-1))

alterH pic1 pic2 n = alt beside pic1 pic2 n

alterV pic1 pic2 n = alt above pic1 pic2 n
```





6.1 Applying functions to all elements of a list: map

Example

```
map even [1, 2, 3]
= [False, True, False]
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map reverse ["abc", "123"]
= ["cba", "321"]
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What is the type of map?

map ::



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map: The mother of all higher-order functions

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Two possible definitions:

$$map f xs = [f x | x < - xs]$$

Predefined in Prelude.
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```
map f xs = [ f x | x <- xs ]

map f [] = []

map f (x:xs) = f x : map f xs
```



Evaluating map

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map f [] = []
map f (x:xs) = f x : map f xs
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map sqr [1, -2]
= map sqr (1 : -2 : [])
= sqr 1 : map sqr (-2 : [])
```



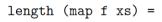
Evaluating map



Some properties of map

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map f [] = []
map f (x:xs) = f x : map f xs

map sqr [1, -2]
= map sqr (1 : -2 : [])
= sqr 1 : map sqr (-2 : [])
```





Some properties of map



Some properties of map



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Some properties of map

length (map f xs) = length xs



Some properties of map



Some properties of map

```
length (map f xs) = length xs
map f (xs ++ ys) = map f xs ++ map f ys
```



Some properties of map



QuickCheck and function variables

length (map f xs) = length xs
map f (xs ++ ys) = map f xs ++ map f ys
map f (reverse xs) = reverse (map f xs)
Proofs by induction

QuickCheck does not work automatically for properties of function variables



QuickCheck and function variables

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It needs to know how to generate and print functions.



QuickCheck and function variables

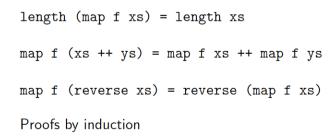
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Cheap alternative: replace function variable by specific function(s)



Some properties of map





QuickCheck and function variables

QuickCheck does not work automatically for properties of function variables

It needs to know how to generate and print functions.

Cheap alternative: replace function variable by specific function(s)

Example

```
prop_map_even :: [Int] -> [Int] -> Bool
prop_map_even xs ys =
  map even (xs ++ ys) = map even xs ++ map even ys
```



6.2 Filtering a list: filter



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Example

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filter even [1, 2, 3] = [2]
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What is the type of filter?
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filter :: (a -> Bool) ->



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= [[], []]
```

What is the type of filter?



filter

filter

Predefined in Prelude.

Two possible definitions:

```
filter p xs = [x | x \leftarrow xs, px]
```

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Some properties of filter

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filter p (xs ++ ys) = filter p xs ++ filter p ys
filter p (reverse xs) = reverse (filter p xs)
Proofs by induction
```