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Chapter 9

Priority Queues

Priority queue informally

Collection of elements with priorities

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Operations:

Priority queue informally

Collection of elements with priorities

Operations:

- empty
- emptiness test
- insert
- get element with minimal priority
- delete element with minimal priority

We focus on the priorities:
element = priority

Priority queues are multisets

The same element can be contained multiple times in a priority queue

Interface of implementation

The type of elements (= priorities) $'a$ is a linear order

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- An implementation type $'q$

More operations

- $merge :: 'q \Rightarrow 'q \Rightarrow 'q$
  Often provided

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- $merge :: 'q \Rightarrow 'q \Rightarrow 'q$
  Often provided
- decrease key/priority
  A bit tricky in functional setting

Correctness of implementation

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Correctness proof requires:

Abstraction function: $mset :: 'q \Rightarrow 'a\ multiset$

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Invariant: $invar :: 'q \Rightarrow bool$

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$mset\ q \neq \{\#\} \Longrightarrow$
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$invar\ empty$
$invar\ (insert\ x\ q)$
$invar\ (del\_min\ q)$

Terminology

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The term "heap" is frequently used synonymously with "priority queue".

Priority queue via heap

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A naive merge:

$merge\ t_1\ t_2 =$ (case (t_1,t_2) of
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Challenge: how to maintain some kind of balance

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Merge descends along the right spine.
Thus rank bounds number of steps.

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Merge descends along the right spine.
Thus rank bounds number of steps.

If rank of right child gets too large: swap with left child.

Implementation type

datatype
$'a\ lheap = Leaf \mid Node\ ('a\ tree)\ 'a\ nat\ ('a\ tree)$

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Abbreviations $\langle\rangle$ and $\langle l,\ a,\ n,\ r\rangle$ as usual

Abstraction function:
$mset\_tree :: 'a\ lheap \Rightarrow 'a\ multiset$
$mset\_tree\ \langle\rangle = \{\#\}$
$mset\_tree\ \langle l,\ a,\ \_,\ r\rangle =$
$\{\#a\#\} + mset\_tree\ l + mset\_tree\ r$

Leftist tree

$rank :: 'a\ lheap \Rightarrow nat$

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Node $\langle l,\ a,\ n,\ r\rangle$: $n =$ rank of node

$ltree :: 'a\ lheap \Rightarrow bool$

$ltree\ \langle\rangle = True$
$ltree\ \langle l,\ \_,\ n,\ r\rangle =$
$(n = rank\ r + 1 \wedge rank\ r \leq rank\ l \wedge ltree\ l \wedge ltree\ r)$

Leftist heap invariant

$invar\ h = (heap\ h \wedge ltree\ h)$

merge

Principle: descend on the right

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$\ \textsf{else}\ node\ l_2\ a_2\ (merge\ t_1\ r_2))$

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15 Priority Queues

16 Leftist Heap

17 Priority Queue via Braun Tree

18 Binomial Heap

19 Skew Binomial Heap

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(if $a_1 \leq a_2$ then $node\ l_1\ a_1\ (merge\ r_1\ t_2)$
else $node\ l_2\ a_2\ (merge\ t_1\ r_2)$)

$node :: 'a\ lheap \Rightarrow 'a \Rightarrow 'a\ lheap \Rightarrow 'a\ lheap$

$node\ l\ a\ r =$
(let $rl = rk\ l$; $rr = rk\ r$
in if $rr \leq rl$ then $\langle l,\ a,\ rr + 1,\ r\rangle$ else $\langle r,\ a,\ rl + 1,\ l\rangle$)
where $rk\ \langle\_,\ \_,\ n,\ \_\rangle = n$

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Function $merge$ terminates because ?
decreases with every recursive call.

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Function *merge* terminates because $size\ t_1 + size\ t_2$ decreases with every recursive call.

Logarithmic complexity

Correlation of rank and size:
Lemma $ltree\ t \Longrightarrow 2^{rank\ t} \le |t|_1$

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$merge$

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Functional correctness proofs

including preservation of $invar$

Straightforward

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Lemma $t\_merge\ l\ r \leq rank\ l + rank\ r + 1$

Corollary $[\![ltree\ l;\ ltree\ r]\!]$
$\Longrightarrow t\_merge\ l\ r \leq \log_2 |l|_1 + \log_2 |r|_1 + 1$

Corollary
$ltree\ t \Longrightarrow t\_insert\ x\ t \leq \log_2 |t|_1 + 2$

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Corollary
$ltree\ t \Longrightarrow t\_del\_min\ t \leq 2 * \log_2 |t|_1 + 1$

What is a Braun tree?

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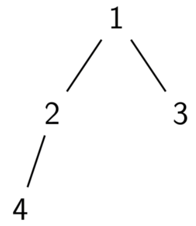

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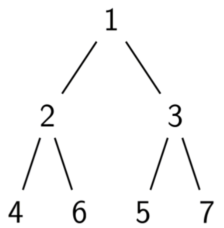

Idea of invariant maintenance

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Add element:

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Add element: to right subtree, then swap subtrees

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Goal: $|l| \leq |r| + 1 \wedge |r| + 1 \leq |l| + 1$

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Priority queue implementation

Implementation type: $'a\ tree$

Invariants: $heap$ and $braun$

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No $merge$

$insert$

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$insert\ a\ \langle\rangle = \langle\langle\rangle,\ a,\ \langle\rangle\rangle$

$insert\ a\ \langle l,\ x,\ r\rangle =$

(if $a < x$ then $\langle insert\ x\ r,\ a,\ l\rangle$ else $\langle insert\ a\ r,\ x,\ l\rangle$)

del_min

$del\_min :: 'a\ tree \Rightarrow 'a\ tree$

del_min

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$del\_min\ \langle\rangle = \langle\rangle$
$del\_min\ \langle\langle\rangle, x, r\rangle = \langle\rangle$
$del\_min\ \langle l, x, r\rangle =$
$(\text{let } (y, l') = del\_left\ l \text{ in } sift\_down\ r\ y\ l')$

del_min

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$del\_min\ \langle l, x, r\rangle =$
$(\text{let } (y, l') = del\_left\ l \text{ in } sift\_down\ r\ y\ l')$

1. Delete leftmost element y
2. Sift y from the root down

Reminiscent of heapsort, but not quite . . .

del_left

$del\_left :: 'a\ tree \Rightarrow 'a \times 'a\ tree$

sift_down

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$sift\_down\ \langle\rangle\ a\ \langle\rangle = \langle\langle\rangle,\ a,\ \langle\rangle\rangle$

$sift\_down\ \langle\langle\rangle,\ x,\ \langle\rangle\rangle\ a\ \langle\rangle =$
$(\text{if } a \le x \text{ then } \langle\langle\langle\rangle,\ x,\ \langle\rangle\rangle,\ a,\ \langle\rangle\rangle$
$\ \text{else } \langle\langle\langle\rangle,\ a,\ \langle\rangle\rangle,\ x,\ \langle\rangle\rangle)$

$sift\_down\ (\langle l_1,\ x_1,\ r_1\rangle =: t_1)\ a\ (\langle l_2,\ x_2,\ r_2\rangle =: t_2) =$
$\text{if } a \le x_1 \wedge a \le x_2 \text{ then } \langle t_1,\ a,\ t_2\rangle$
$\text{else if } x_1 \le x_2 \text{ then } \langle sift\_down\ l_1\ a\ r_1,\ x_1,\ t_2\rangle$
$\quad \text{else } \langle t_1,\ x_2,\ sift\_down\ l_2\ a\ r_2\rangle$

sift_down

$sift\_down :: 'a\ tree \Rightarrow 'a \Rightarrow 'a\ tree \Rightarrow 'a\ tree$

$sift\_down\ \langle\rangle\ a\ \langle\rangle = \langle\langle\rangle,\ a,\ \langle\rangle\rangle$

$sift\_down\ \langle\langle\rangle,\ x,\ \langle\rangle\rangle\ a\ \langle\rangle =$
$(\text{if } a \le x \text{ then } \langle\langle\langle\rangle,\ x,\ \langle\rangle\rangle,\ a,\ \langle\rangle\rangle$
$\ \text{else } \langle\langle\langle\rangle,\ a,\ \langle\rangle\rangle,\ x,\ \langle\rangle\rangle)$

del_left

$del\_left :: 'a\ tree \Rightarrow 'a \times 'a\ tree$

Functional correctness proofs for *del_min*

Many lemmas, mostly straightforward

Logarithmic complexity

Running time of *insert*, *del_left* and *sift_down* (and therefore *del_min*) bounded by height

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Running time of *insert*, *del_left* and *sift_down* (and therefore *del_min*) bounded by height

Remember: $braun\ t \Longrightarrow 2^{h(t)} \leq 2 * |t| + 1$

Source of code

Based on code from
L.C. Paulson. *ML for the Working Programmer.* 1996
based on code from Chris Okasaki.

Sorting with priority queue

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$sort\_pq = mins \circ pq$

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$mins\ q =$
(if $is\_empty\ q$ then $[]$
else $get\_min\ h \mathbin{\#} mins\ (del\_min\ h))$

$sort\_pq = mins \circ pq$

Complexity of $sort$: $O(n \log n)$
if all priority queue functions have complexity $O(\log n)$

15 Priority Queues

16 Leftist Heap

17 Priority Queue via Braun Tree

18 Binomial Heap

19 Skew Binomial Heap