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Using equations $l = r$ from left to right

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Terminology: equation $\leadsto$ *simplification rule*

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Simplification = (Term) Rewriting

An example

Equations:

$$\begin{array}{rcll} 0 + n & = & n & (1) \\ (Suc\ m) + n & = & Suc\ (m + n) & (2) \\ (Suc\ m \leq Suc\ n) & = & (m \leq n) & (3) \\ (0 \leq m) & = & True & (4) \end{array}$$

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$$\begin{array}{rcl} p(0) & = & True \\ p(x) \Longrightarrow f(x) & = & g(x) \end{array}$$

We can simplify $f(0)$ to $g(0)$ but we cannot simplify $f(1)$ because $p(1)$ is not provable.

Termination

Simplification may not terminate.
Isabelle uses $simp$-rules (almost) blindly from left to right.

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$n < m \Longrightarrow (n < Suc\ m) = True$ YES
$Suc\ n < m \Longrightarrow (n < m) = True$ NO

Proof method *simp*

Goal: 1. $[\![\ P_1; \ldots; P_m\ ]\!] \Longrightarrow C$

apply(*simp add:* $eq_1 \ldots eq_n$)

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- rules from **fun** and **datatype**

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Variations:

- (*simp* ... *del:* ...) removes *simp*-lemmas
- *add* and *del* are optional

auto versus *simp*

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- *auto* applies *simp* and more
- *auto* can also be modified:
  $(auto\ simp\ add\colon \ldots\ simp\ del\colon \ldots)$

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- *auto* acts on all subgoals
- *simp* acts only on subgoal 1

Rewriting with definitions

Definitions (**definition**) must be used explicitly:

$$(simp\ add{:}\ f\_def\ \ldots)$$

Case splitting with *simp/auto*

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$$P\ (\text{case } e \text{ of } 0 \Rightarrow a \mid Suc\ n \Rightarrow b) \\ = \\ (e = 0 \longrightarrow P(a)) \wedge (\forall n.\ e = Suc\ n \longrightarrow P(b))$$

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Proof method: $(simp\ split\colon\ nat.split)$
Or $auto$. Similar for any datatype t: $t.split$

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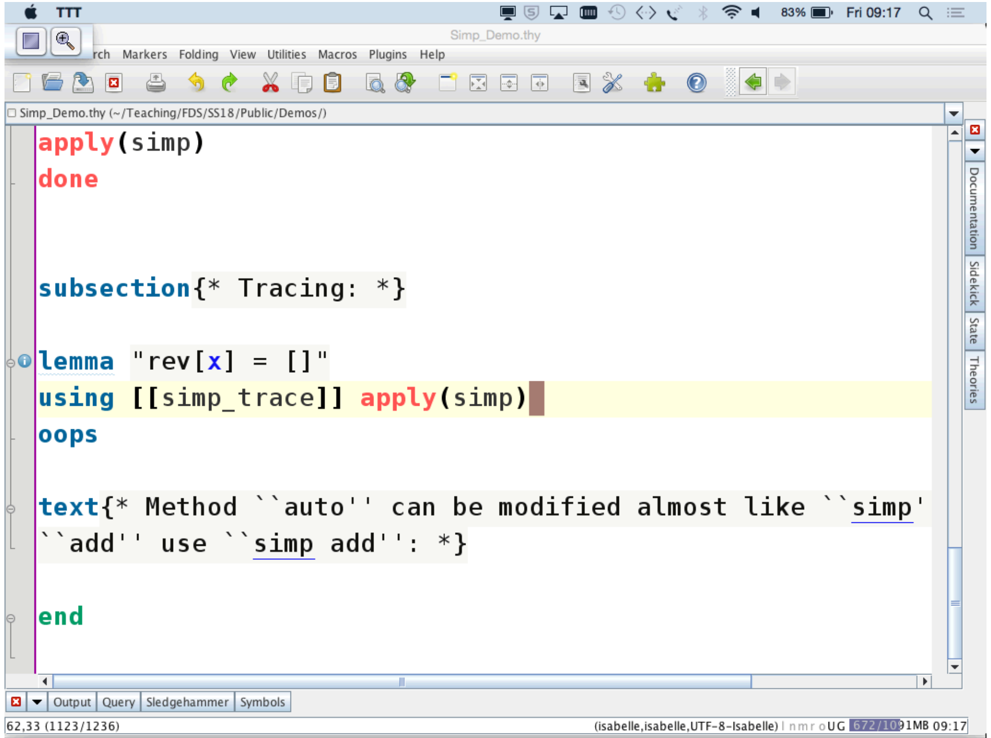
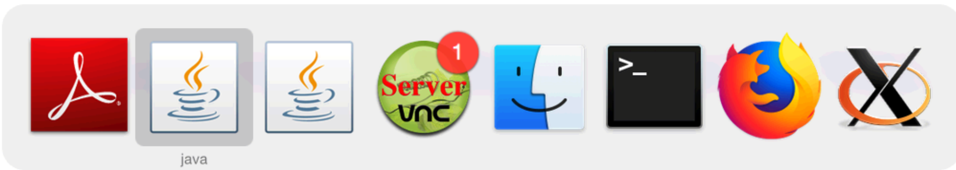
Proof method: *(simp split: prod.split)*

Simp_Demo.thy



Chapter 3



Preview: sets

Type: $'a\ set$

Operations: $a \in A$, $A \cup B$, ...

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Proof method *auto* knows (a little) about sets.

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Example: $bst :: ('a :: linorder)\ tree \Rightarrow bool$

Case study

`BST_Demo.thy`

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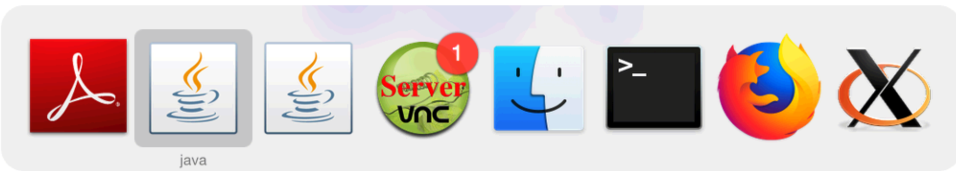

Chapter 4

Logic and Proof Beyond Equality