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Chapter 4

Logic and Proof Beyond Equality

5 Logical Formulas

6 Proof Automation

7 Single Step Proofs

Syntax (in decreasing precedence):

$$
\begin{array}{rcllll}
\mathit{form} & ::= & (\mathit{form}) & \mid\ \mathit{term} = \mathit{term} & \mid\ \neg\mathit{form} \\
& \mid & \mathit{form} \wedge \mathit{form} & \mid\ \mathit{form} \vee \mathit{form} & \mid\ \mathit{form} \longrightarrow \mathit{form} \\
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Input syntax: $\longleftrightarrow$ (same precedence as $\longrightarrow$)

Variable binding convention:

$$\forall\, x\ y.\ P\ x\ y \ \equiv\ \forall\, x.\ \forall\, y.\ P\ x\ y$$

Warning

Quantifiers have low precedence
and need to be parenthesized (if in some context)

! $P \wedge \forall\, x.\ Q\ x \ \leadsto\ P \wedge (\forall\, x.\ Q\ x)$ **!**

Mathematical symbols

... and their ascii representations:

<table>
<tr><td>∀</td><td>\<forall></td><td>ALL</td></tr>
<tr><td>∃</td><td>\<exists></td><td>EX</td></tr>
<tr><td>λ</td><td>\<lambda></td><td>%</td></tr>
<tr><td>⟶</td><td>--></td><td></td></tr>
<tr><td>⟷</td><td><-></td><td></td></tr>
<tr><td>∧</td><td>/\</td><td>&</td></tr>
<tr><td>∨</td><td>\/</td><td>|</td></tr>
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- ...

$\in$	`\<in>`	`:`
$\subseteq$	`\<subseteq>`	`<=`
$\cup$	`\<union>`	`Un`
$\cap$	`\<inter>`	`Int`

simp and *auto*

simp: rewriting and a bit of arithmetic

auto: rewriting and a bit of arithmetic, logic and sets

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- Extensible with new *simp*-rules

Exception: *auto* acts on all subgoals

fastforce

- rewriting, logic, sets, relations and a bit of arithmetic.

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- rewriting, logic, sets, relations and a bit of arithmetic.

- incomplete but better than *auto*.
- Succeeds or fails

blast

- A complete proof search procedure for FOL ...

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- ... but (almost) without "="

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- A complete proof search procedure for FOL ...
- ... but (almost) without "="
- Covers logic, sets and relations
- Succeeds or fails

Sledgehammer

hitecture:

Isabelle

external
ATPs[1]

[1] Automatic Theorem Provers

hitecture:

Isabelle

Goal
& filtered library ↓

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ATPs[1]

Characteristics:

- Sometimes it works,
- sometimes it doesn't.

Do you feel lucky?

[1] Automatic Theorem Provers

by($proof\text{-}method$)

$\approx$

apply($proof\text{-}method$)
done

❻ Proof Automation
Automating Arithmetic

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Linear formulas

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number $*$ variable

$+, -$

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$\neg, \wedge, \vee, \longrightarrow, \longleftrightarrow$

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Examples

Linear: $3 * x + 5 * y \leq z \longrightarrow x < z$

Extended linear formulas

Also allowed:

min, *max*

even, *odd*

t *div* n, t *mod* n where n is a number

conversion functions

nat, *floor*, *ceiling*, *abs*

Automatic proof of arithmetic formulas

by *arith*

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Proof method *arith* tries to prove arithmetic formulas.

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- Decision procedure for extended linear formulas

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Proof method *arith* tries to prove arithmetic formulas.

- Succeeds or fails
- Decision procedure for extended linear formulas
- Nonlinear subterms are viewed as (new) variables.
  Example: $x \le x * x + f\ y$ is viewed as $x \le u + v$

Automatic proof of arithmetic formulas

by $(simp\ add{:}\ algebra\_simps)$

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- It contains associativity, commutativity and distributivity of $+$ and $*$.

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- The lemmas list *field_simps* extends *algebra_simps* by rules for /
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Example

simp add: numeral_eq_Suc rewrites x ^ 2 to $x * x$

`Auto_Proof_Demo.thy`

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- By unification:
  unifying $?P \wedge ?Q$ with $a{=}b \wedge False$

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The general case: applying rule $[\![A_1; \ldots ; A_n]\!] \Longrightarrow A$ to subgoal $\ldots \Longrightarrow C$:

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"Backchaining"

Typical backwards rules

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$$\frac{?P \Longrightarrow ?Q \quad ?Q \Longrightarrow ?P}{?P = ?Q}\ \texttt{iffI}$$

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$$\rightsquigarrow$$

$$?Q \Longrightarrow a = a \wedge ?Q$$

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$$\leadsto$$

$$a = a \land b = b$$

From now on: $?$ mostly suppressed on slides

Single_Step_Demo.thy

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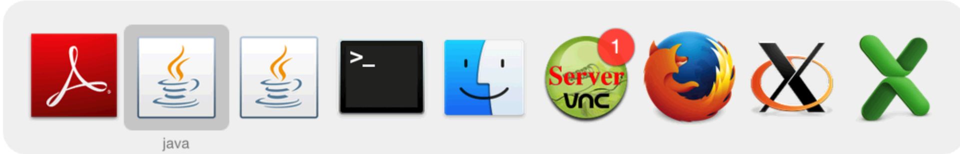

Case distinction

show R
proof *cases*
  assume P
  ⋮
  show R ...
next
  assume $\neg P$
  ⋮
  show R ...
qed

have $P \vee Q$ ...
then show R
proof
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  ⋮
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